Supplemental material for article “Neural Networks as Geometric Chaotic Maps”

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Abstract. This file contains the supplementary figures and texts for article ”Neural Networks as Geometric Chaotic Maps”. We describe numerical computation of FTLE and the connection between the geometric map and the original Lorenz map, and lay out supplementary figures.

S1. Numerical computation of FTLE

We define the mapping over N_t time steps as $\Phi_{0\to N_t} : \mathbf{x}_0 \mapsto \mathbf{x}_{N_t}$. When a perturbation $\delta\mathbf{x}$ around $\mathbf{x}_0$ is sufficiently small and N_t is finite, the resulting distance $\delta\mathbf{x}_{N_t}$ can be linearly approximated as

$$\delta\mathbf{x}_{N_t} = \mathbf{J}_{N_t}(\mathbf{x}_0)\delta\mathbf{x} + O(|\delta\mathbf{x}|^2), \tag{S1}$$

where $\mathbf{J}_{N_t}(\mathbf{x}_0)$ is the Jacobian of mapping $\Phi_{0\to N_t}$ evaluated by forward-propagating perturbations around $\mathbf{x}_0$ for N_t steps. Neglecting higher-order terms in (S1) and taking its norm, we have

$$|\delta\mathbf{x}_{N_t}|^2 = \delta\mathbf{x}^{\mathrm{T}}\mathbf{J}_{N_t}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{J}_{N_t}(\mathbf{x}_0)\delta\mathbf{x}, \tag{S2}$$

Then, the problem of finding the direction of $\mathbf{x}_0$ to maximize the growth rate of perturbation reduces to solving for the eigenvector that corresponds to the largest eigenvalue of matrix $\mathbf{J}_{N_t}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{J}_{N_t}(\mathbf{x}_0)$, and the largest growth rate corresponds to its largest eigenvalue, σ_{max}. The maximum FTLE is therefore evaluated as

$$\lambda_{\text{max}} = \frac{1}{N_t}\ln\sqrt{\sigma_{\text{max}}}, \tag{S3}$$

where $\sigma_{\max}$ is the largest eigenvalue of $\mathbf{J}_{N_t}(\mathbf{x}_0)^{\mathrm{T}}\mathbf{J}_{N_t}(\mathbf{x}_0)$. (S3) is used to calculate the local maximum FTLE in section III-B.

S2. Exact correspondence between the geometric Lorenz map and L63

The fact that L63 system is a first-order ODE set and cannot be analytically solved means an exact solution can never be obtained. Fortunately, ref. [1] rigorously proved that the numerical solution of the original dynamical equations has the same topological properties as the *geometric Lorenz flow* proposed by [2]. The geometric flow has a compressing operation in the x direction and a stretching mainly in the y-z plane (Fig. S1). The stretching has two important properties: first, it has an anti-symmetric x component such that the surfaces S_1 and S_2, which were originally separated in the y direction are now separated in x; second, it stretches in the y direction such that S_1'' and S_2'' becomes *mixed* on the original S_1 and S_2 manifold. These geometric operations are such that a horseshoe attractor emerges after a number of iterations, leading to topological mixing and chaos.

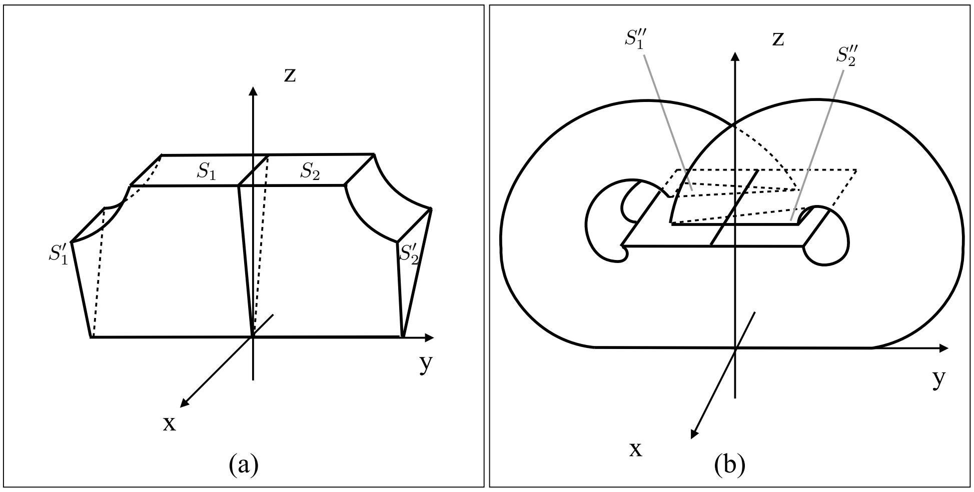

Figure S1: Schematic of the geometric Lorenz flow, similar to Figs. 1 and 2 in [2]. (a) A rectangular S parallel to the x-y plane at the positive z axis is divided into S_1 ($y < 0$) and S_2 ($y > 0$). S_1 is then moved to the lower left, compressed in x and becomes triangle S_1', whereas S_2 is moved symmetrically to the right and becomes S_2'. (b) S_1' and S_2' are swirled and mapped back onto the original rectangular as S_1'' and S_2'', respectively. S_1'' satisfies $x < 0$, and S_2'' satisfies $x > 0$, so that they occupy mutually exclusive regions. After multiple iterations of this geometric flow, two horseshoe attractors will emerge on S. Note that the directions of x and y are rotated 45° counter-clockwise compared to the original L63 system for ease of illustration.

S3. Supplementary figures

References

[1] W. Tucker, "A rigorous ODE solver and smale's 14th problem," *Foundations of Computational Mathematics*, vol. 2, pp. 53–117, 2002.

[2] J. Guckenheimer and R. F. Williams, "Structural stability of Lorenz attractors," *Publ. Math. IHES*, vol. 50, pp. 307–320, 1979.

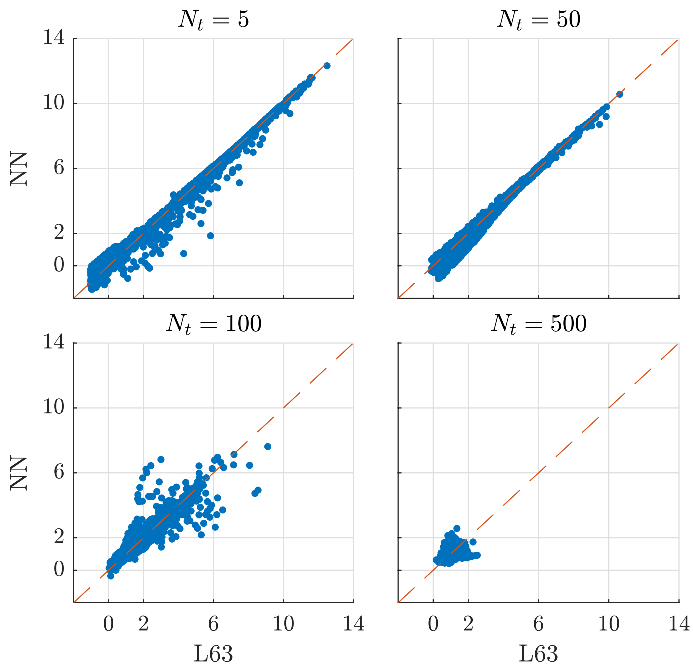

Figure S2: Similar to Fig. 2, but for the 5-neuron NN trained on 100 data points sampled from the $X > -5$ part of the L63 attractor.

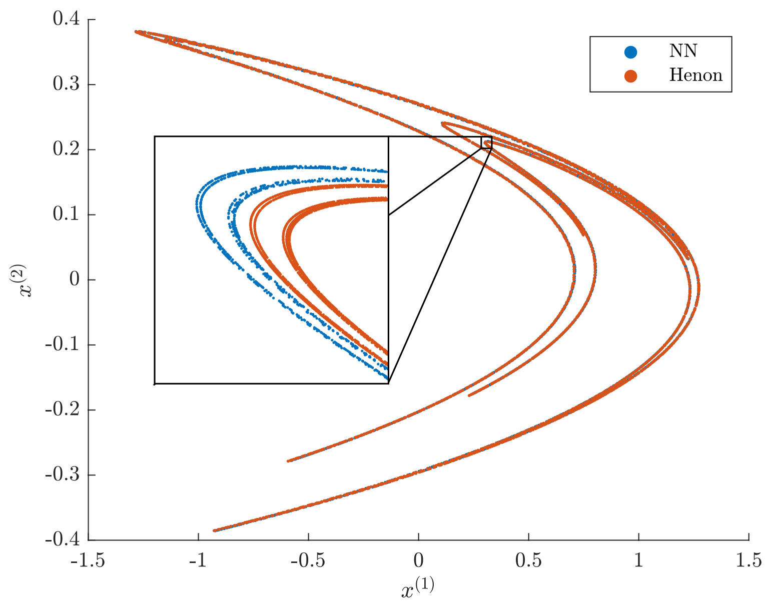

Figure S3: Reconstructed attractor of the 2-neuron toy model. The inset shows a magnified region of $[0.295, 0.313] \times [0.206, 0.214]$, from which a fractal structure can be seen.